

Asymptotic density of Catalan numbers modulo 3 and powers of 2

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Abstract

We establish the asymptotic density of the Catalan numbers modulo 3 and modulo 2^k for $k \in \mathbb{N}$ and $k \geq 1$.

1 Introduction

The *Catalan numbers* are defined by

$$C_n := \frac{1}{n+1} \binom{2n}{n}.$$

There has been much work in recent years and also going back to Kummer [6] on analysing the Catalan numbers modulo primes and prime powers. Deutsch and Sagan [2] provided a complete characterisation of Catalan numbers modulo 3. A characterisation of the Catalan numbers modulo 2 dates back to Kummer. Eu, Liu and Yeh [3] provided a complete characterisation of Catalan numbers modulo 4 and 8. This was extended by Liu and Yeh [8] to a complete characterisation modulo 8, 16 and 64. This result was restated in a more compact form by Kauers, Krattenthaler and Müller in [4] by representing the generating function of C_n as a polynomial involving a special function. The polynomial for C_n modulo 4096 was also calculated. A method for extracting the coefficients of the generating function (i.e. C_n modulo a prime power) was provided, though given the complexity of the polynomials (the polynomial for the 4096 case takes a page and a half to write down) this would need to be done by computer. Krattenthaler and Müller [5] used a similar method to examine C_n modulo powers of 3. They wrote down the polynomial for the generating function of C_n modulo 9 and 27 thereby generalised the mod 3 result of [2]. The article by Lin [7] discussed the possible values of the odd Catalan numbers modulo 2^k and Chen and Jiang [1] dealt with the possible values of the Catalan numbers modulo prime powers. Rowland and Yassawi [9] investigated C_n in the general setting of

automatic sequences. The values of C_n (as well as other sequences) modulo prime powers can be computed via automata. Rowland and Yassawi provided algorithms for creating the relevant automata. They established a full characterisation of C_n modulo $\{2, 4, 8, 16, 3, 5\}$ in terms of automata. They also extended previous work by establishing forbidden residues for C_n modulo $\{32, 64, 128, 256, 512\}$. In theory the automata can be constructed for any prime power but computing power and memory quickly becomes a barrier.

Some of this work can be used to determine the asymptotic densities of the Catalan numbers modulo 2^k and 3.

In section 2 we will use a result of Liu and Yeh [8] to obtain some asymptotic densities of the Catalan numbers modulo powers of 2. Here, the asymptotic density of a subset S of \mathbb{N} is defined to be

$$\lim_{N \rightarrow \infty} \frac{1}{N} \# \{n \in S : n \leq N\}$$

if the limit exists, where $\#S$ is the number of elements in a set S . Since the Catalan numbers C_n are highly multiplicative as n increases, it is expected that for a fixed $m \in \mathbb{N}$ “almost all” Catalan numbers are divisible by m . We show that this is the case when $m = 3$ and $m = 2^k$ for $k \geq 1$.

2 Catalan numbers modulo 2^k

Firstly, as in [8], let the p -adic order of a positive integer n be defined by

$$\omega_p(n) := \max\{\alpha \in \mathbb{N} : p^\alpha | n\}$$

and the cofactor of n with respect to $p^{\omega_p(n)}$ be defined by

$$CF_p(n) := \frac{n}{p^{\omega_p(n)}}.$$

In addition the function $\alpha : \mathbb{N} \rightarrow \mathbb{N}$ is defined by

$$\alpha(n) := \frac{CF_2(n+1) - 1}{2}.$$

For a number p , we write the base p expansion of a number n as

$$[n]_p = \langle n_r n_{r-1} \dots n_1 n_0 \rangle$$

where $n_i \in [0, p-1]$ and

$$n = n_r p^r + n_{r-1} p^{r-1} + \dots + n_1 p + n_0.$$

Then the function $d_p : \mathbb{N} \rightarrow \mathbb{N}$ is defined by

$$d_p(n) := \#\{i : n_i = 1\}. \quad (1)$$

Here we will only be interested in the case $p = 2$ and will refer to d_2 as merely d . When $p = 2$, $d(n)$ is the sum of the digits in the base 2 representation of n .

The following result appears in [8].

Theorem 1 (*Corollary 4.3 of [8]*). *In general, we have $\omega_2(C_n) = d(\alpha(n))$. In particular, $C_n \equiv 0 \pmod{2^k}$ if and only if $d(\alpha(n)) \geq k$, and $C_n \equiv 2^{k-1} \pmod{2^k}$ if and only if $d(\alpha(n)) = k - 1$.*

Since we are looking at the Catalan numbers $C_n \pmod{2^k}$ it will be convenient to first consider numbers $n < 2^t$ for a fixed but arbitrary $t \in \mathbb{N}$. This produces some interesting and simple formulae. We have the following results:

Theorem 2. $\#\{n < 2^t : d(\alpha(n)) = k\} = \binom{t}{k+1}$.

Proof. Let

$$\alpha = \alpha(n) = \frac{CF_2(n+1) - 1}{2}.$$

Then $n+1 = 2^s(2\alpha+1)$ for some arbitrary $s \in \mathbb{N} : s \geq 0$. Writing $n+1$ in base 2 we have

$$[n+1]_2 = \langle [\alpha]_2, 1, 0\dots 0, 0 \rangle$$

where there are s 0's at the end and $s \geq 0$ is arbitrary. So,

$$[n]_2 = \langle [\alpha]_2, 0, 1\dots 1, 1 \rangle$$

where there are s 1's at the end and $s \geq 0$ is arbitrary. It can be seen that $d(n) = d(\alpha) + s$. Since $n < 2^t$ and $d(\alpha(n)) = k$ the possible base 2 representations of n are

$$[n]_2 = \langle [\alpha]_2, 0 \rangle : \alpha < 2^{t-1}$$

$$[n]_2 = \langle [\alpha]_2, 0, 1 \rangle : \alpha < 2^{t-2}$$

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$$[n]_2 = \langle [\alpha]_2, 0, 1, 1, \dots 1 \rangle : \alpha < 2^k$$

and there are $(t - k - 1)$ 1's at the end of the last representation.

Therefore, counting each of these possibilities and using the fact that $d(\alpha) = k$ gives

$$\#\{n < 2^t : d(\alpha(n)) = k\} = \sum_{i=k}^{t-1} \binom{i}{k} = \binom{t}{k+1}.$$

□

Corollary 3. $\#\{n < 2^t : C_n \equiv 0 \pmod{2^k}\} = \sum_{i=k+1}^{i=t} \binom{t}{i}$.

Proof. The corollary follows from Theorem 1 and Theorem 2 above since

$$\begin{aligned}
& \#\{n < 2^t : C_n \equiv 0 \pmod{2^k}\} \\
&= \#\{n < 2^t : d(\alpha(n)) \geq k\} \text{ from Theorem 1} \\
&= 2^t - 1 - \sum_{i=0}^{k-1} \#\{n < 2^t : d(\alpha(n)) = i\} \\
&= 2^t - 1 - \sum_{i=0}^{k-1} \binom{t}{i+1} \text{ from Theorem 2} \\
&= 2^t - \sum_{i=0}^k \binom{t}{i} \\
&= \sum_{i=k+1}^t \binom{t}{i} \text{ since } 2^t = \sum_{i=0}^t \binom{t}{i}
\end{aligned}$$

□

Corollary 4. $\#\{n < 2^t : C_n \equiv 2^{k-1} \pmod{2^k}\} = \binom{t}{k}$.

Proof. From Theorem 1,

$$\begin{aligned}
& \#\{n < 2^t : C_n \equiv 2^{k-1} \pmod{2^k}\} \\
&= \#\{n < 2^t : d(\alpha(n)) = k-1\} \\
&= \binom{t}{k} \text{ from Theorem 2.}
\end{aligned}$$

□

Theorem 2 can be used to establish the asymptotic density of the set

$$\{n < N : C_n \equiv 0 \pmod{2^k}\}$$

Theorem 5. For $k \in \mathbb{N}$ with $k \geq 1$ we have

$$\lim_{N \rightarrow \infty} \frac{1}{N} \#\{n < N : C_n \equiv 0 \pmod{2^k}\} = 1.$$

Proof. Let $N \in \mathbb{N}$ and choose $r \in \mathbb{N}$ such that

$$2^r \leq N < 2^{r+1}$$

Then $r = \lfloor \log_2(N) \rfloor$ and

$$\begin{aligned} & \#\{n < N : d(\alpha(n)) = k\} \\ & \leq \#\{n < 2^{r+1} : d(\alpha(n)) = k\} \\ & = \binom{r+1}{k+1}. \end{aligned}$$

So,

$$\begin{aligned} & \#\{n < N : C_n \equiv 0 \pmod{2^k}\} \\ & = N - \#\{n < N : d(\alpha(n)) < k\} \\ & = N - \sum_{i=0}^{k-1} \#\{n < N : d(\alpha(n)) = i\} \\ & \geq N - \sum_{i=0}^{k-1} \binom{r+1}{i+1}. \end{aligned}$$

And so,

$$\begin{aligned} & \frac{1}{N} \#\{n < N : C_n \equiv 0 \pmod{2^k}\} \\ & \geq 1 - \frac{1}{N} P_k(r) \\ & = 1 - \frac{1}{N} P_k(\lfloor \log_2(N) \rfloor) \end{aligned} \tag{2}$$

where $P_k(r)$ is a polynomial in r of degree k with coefficients depending on k . Since k is fixed and

$$\lim_{N \rightarrow \infty} \frac{1}{N} (\lfloor \log_2(N) \rfloor)^k = 0$$

for fixed k , the second term in (2) is zero in the limit and

$$\lim_{N \rightarrow \infty} \frac{1}{N} \#\{n < N : C_n \equiv 0 \pmod{2^k}\} = 1.$$

□

3 Catalan numbers modulo 3

Deutsch and Sagan [2] provided a characterisation of the Catalan numbers modulo 3. They defined a set $T^*(01)$ of natural numbers using the base 3 representation. If $[n]_3 = \langle n_i \rangle$ is the base 3 representation of n then

$$T^*(01) = \{n \geq 0 : n_i = 0 \text{ or } 1 \text{ for all } i \geq 1\}.$$

They then defined the function $d_3^*(n)$ similar to $d_p(n)$ in (1) as

$$d_3^*(n) = \#\{n_i : i \geq 1, n_i = 1\}$$

Theorem 6. *The asymptotic density of the set $T^*(01)$ is zero.*

Proof. Let $N \in \mathbb{N}$ and choose $k \in \mathbb{N} : 3^k \leq N < 3^{k+1}$. Then $k = \lfloor \log_3(N) \rfloor$ and

$$\begin{aligned} \frac{1}{N} \#\{n \leq N : n \in T^*(01)\} \\ \leq \frac{1}{N} 3 \times 2^k \\ \leq 3^{1-k} \times 2^k \rightarrow 0 \text{ as } N \rightarrow \infty. \end{aligned}$$

□

The following theorem comes from [2].

Theorem 7. *(Theorem 5.2 of [2]) The Catalan numbers satisfy*

$$\begin{aligned} C_n &\equiv (-1)^{d_3^*(n+1)} \pmod{3} \text{ if } n \in T^*(01) - 1 \\ C_n &\equiv 0 \pmod{3} \text{ otherwise.} \end{aligned}$$

Corollary 8.

$$\lim_{N \rightarrow \infty} \frac{1}{N} \#\{n < N : C_n \equiv 0 \pmod{3}\} = 1.$$

Proof. Combining theorems 6 and 7, the set of n such that C_n is not congruent to 0 mod 3 has asymptotic density 0. □

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